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The role of periodic solutions in the Falkner-Skan problem for $\lambda > 0$

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Summary

The Falkner-Skan equation $f''' + ff'' + \lambda(1 - f'^2) = 0$ is discussed for $\lambda > 0$. Two types of solutions have been pursued: those satisfying $f(0) = f'(0) = 0$, $f'(\infty) = 1$ and those being periodic. In both cases, numerical evidence is given for a rich structure of multiple solutions. Branching occurs for $\lambda = 1, 2, 3, \ldots$. All solutions can be characterized by means of a special subset of periodic solutions.

1. Introduction

More than half a century ago Falkner and Skan [1] introduced a class of similarity solutions in boundary-layer theory, governed by the differential equation

$$f''' + ff'' + \lambda(1 - f'^2) = 0 \tag{1}$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1. \tag{2}$$

Although since then this equation has been studied extensively, we will show that it has not yet revealed all of its secrets.

Let us first summarize the properties of (1) + (2) which are known so far.

- For $\lambda < -1$ an infinite number of solutions exists in which $f' \to 1$ exponentially. These solutions, which exhibit overshoot (i.e. $f' > 1$ for some values of the argument η), emanate from a giant branching point at $\lambda = -1$, $f''(0) = -1.08638$; see Oskam and Veldman [2].
- For $-1 \leqslant \lambda < \lambda^* = -0.1988$ no exponentially decaying solutions exist.
- For $\lambda^* < \lambda < 0$ Hastings [3] has proved the existence and uniqueness of two solutions of (1) + (2): one with $f''(0) > 0$, the other one with $f''(0) < 0$.
- For $0 \leqslant \lambda \leqslant 1$ (1) + (2) possesses a unique solution, as shown by Coppel [4] and Craven and Peletier [5].
- For $\lambda > 1$ the solution is unique under the restriction $0 < f' < 1$; see, for instance, Hartman [6]. Solutions not satisfying this restriction have been presented by Craven and Peletier [7].

The solutions obtained by Craven and Peletier exhibit a regular behaviour, strongly suggesting that for $\lambda > 1$ a periodic solution exists as in the case $\lambda < -1$ [2]. The present paper pursues this possibility. Numerical evidence is supplied that there is even a multitude of periodic solutions. This is shown to be related to a complex branching structure of solutions of (1) + (2). For $1 < \lambda \leqslant 2$ the solutions of Craven and Peletier appear to be the only ones, but at $\lambda = 2$ a bifurcation occurs, leading to additional solutions. The bifurcation process repeats itself at every next positive integer value of λ.

2. Numerical solutions of (1) + (2)

In order to find numerical solutions of (1) + (2), we consider (1) together with the initial conditions

$$f(0) = f'(0) = 0, \quad f''(0) = \tau \tag{3}$$

as an initial-value problem. The results of Craven and Peletier [7] suggest that for $\lambda > 1$ multiple solutions of (1) + (2) exist with 0, 1, 2,... relative minima for f'. If we denote the respective values of $f''(0)$ by $\tau_0, \tau_1, \tau_2, \ldots,$ their results also suggest the following ordering:

$$\tau_1 < \tau_3 < \tau_5 < \ldots < \tau_4 < \tau_2 < \tau_0. \tag{4}$$

Thus, increasing $f''(0)$ from τ_1 to τ_3, a solution with one minimum is deformed into a solution with three minima. Being intrigued by this process and keeping in mind the rich structure for negative values of λ, we decided to consider the case $\lambda > 1$ in detail. As a start we studied the dependence of the solution of the initial-value problem (1) + (3) on the value τ of $f''(0)$, using an accurate numerical integration method of the extrapolation type as described by Bulirsch and Stoer [8].

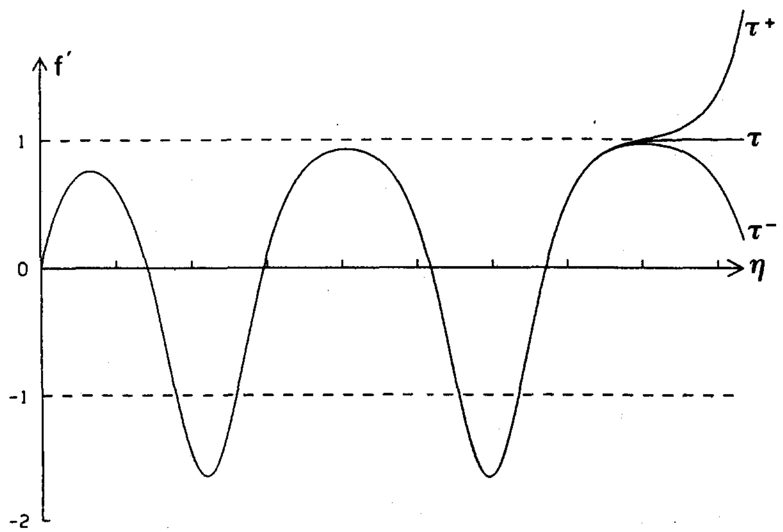

Figure 1. Behaviour of a Falkner-Skan solution with $f''(0) = \tau$ and the integral curves with $f''(0) = \tau^{\pm}$.

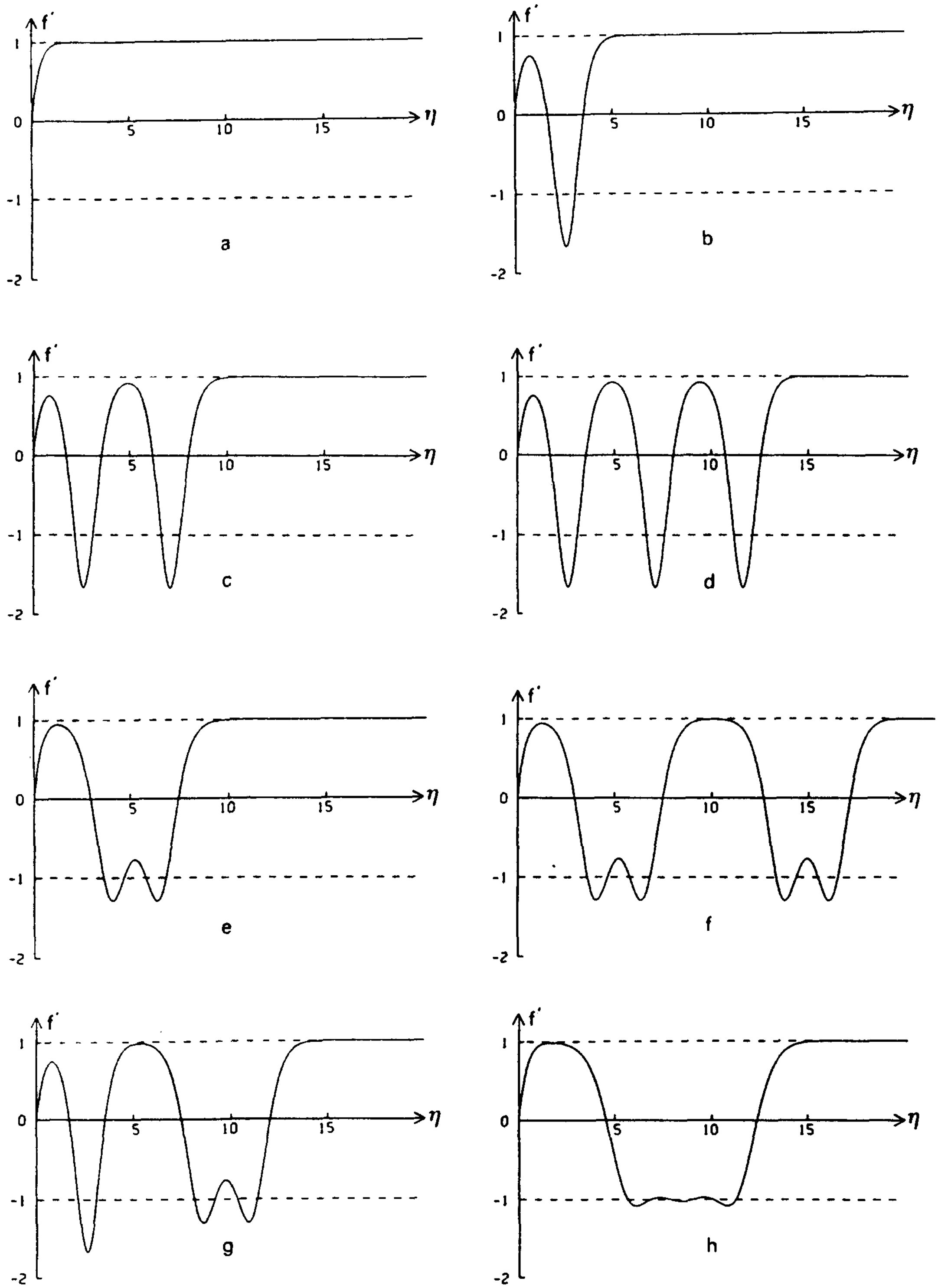

Figure 2. Some Falkner-Skan solutions for $\lambda = 4$.

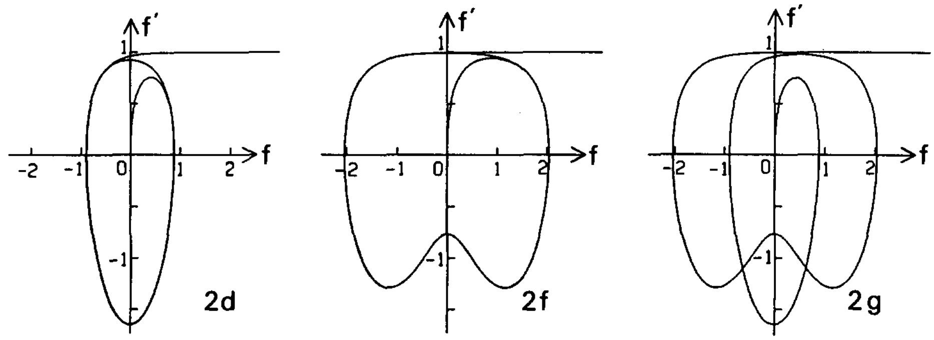

Figure 3. Three solutions from Fig. 2 in the (f, f')-plane.

When trying to locate solutions f of (1) + (2), the following remarks concerning its derivative f' are of importance:

(i) at the (relative) maxima of f' we have $|f'| < 1$;
(ii) at the (relative) minima of f' we have $f' < -1$;
(iii) $1 - f' \approx c_0\eta^{-1-2\lambda} \exp(-\frac{1}{2}\eta^2 - c_1\eta)$ as $\eta \to \infty$ $(c_0 > 0)$.

Remark (i) follows straightforwardly by considering the sign of f''' for $f'' = 0$. Similarly, for the relative minima it follows that $|f'| > 1$, but because of (i) and $f'(\infty) = 1$, f' can never exceed 1. For remark (iii) we refer to the work of Coppel [4] and Hartman [9]. From these remarks we see that f' oscillates around -1. Moreover, the limit $f' = 1$ is approached from below in a fast exponential way and the numerical integration of (1) can be stopped as soon as f' becomes greater than 1.

Scanning the values of $f''(0)$ it became clear that a solution f of (1) + (2) and its value τ of $f''(0)$ can be well characterized by the behaviour in the (η, f')-plane of the integral curves for adjacent values $\tau^{\pm}$. A disturbance to the one side leads to overshoot, whereas a disturbance to the other side leads to a new relative maximum for f', close to 1. Figure 1 gives a typical situation, but the role of τ^+ and τ^- may be interchanged.

With a shooting algorithm based upon this characterization we encountered for $\lambda = 4$ the solutions given in Fig. 2. The first four solutions (a–d) are of the Craven and Peletier type, but the others are new in the sense that they possess negative relative maxima of f'. The regular behaviour of the solutions d and f strongly suggests the existence of periodic solutions, just as in the case $\lambda < -1$, c.f. [2]. We are strenghtened in this opinion when considering Fig. 3, where the solutions d, f and g are given in the (f, f')-plane. It looks as if the suggested periodic solutions of d and f are combined in the solution g, which raises the question of a possible classification in terms of periodic solutions.

3. Periodic solutions of (1) for $\lambda > 1$

If $f(\eta)$ satisfies (1), it is evident that also $-f(-\eta)$ is a solution of (1). Hence, looking for a periodic solution of (1), it is natural to consider first the possibility of antisymmetric periodic solutions, which were also encountered for $\lambda < -1$ in [2]. No periodic solutions

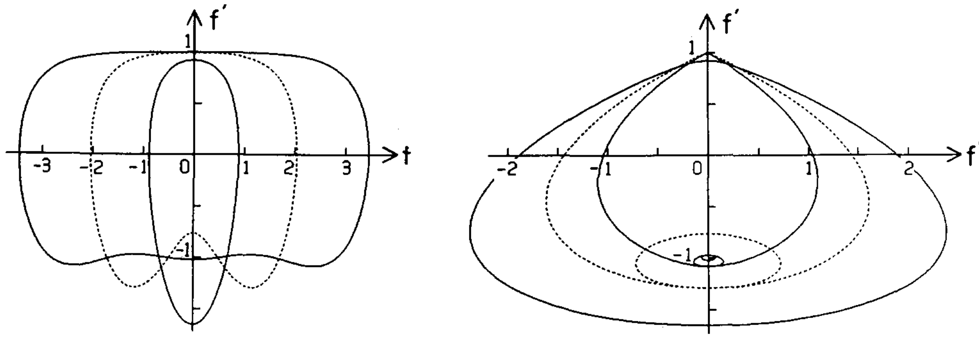

Figure 4. Three periodic solutions of (1) for $\lambda = 4$.

exist for $0 \leqslant \lambda \leqslant 1$, as proved by Coppel [4]. The existence of a periodic solution for $-1 \leqslant \lambda < 0$ can easily be contradicted by integrating (1) over its period.

To calculate antisymmetric periodic solutions for $\lambda > 1$, we start with $f(0) = f''(0)$ and again use a shooting algorithm to determine $f'(0)$ in such a way that f'' becomes zero in a next zero of f. This approach turned out to be quite successful and Fig. 4 shows for $\lambda = 4$ three resulting periodic solutions in the (f, f')-plane and the (f'', f')-plane. We will denote them by P-i, where i is the number of (relative) minima of f' for one period. With a similar calculation for $\lambda = 6$ we also produced periodic solutions of type P-4 and P-5, but for $1 < \lambda \leqslant 2$ only type P-1 was found. Therefore we decided to follow P-1, P-2 and P-3 with decreasing λ. A survey of $f'(0)$ values and the corresponding periods is given in Table 1. These results clearly illustrate the process of disappearing of the periodic solutions of type P-i when λ approaches i from above: the period grows and $f'(0)$ tends to 1. For $\lambda \leqslant i$ type P-i has disappeared. In Fig. 5 this process can be followed for type P-2 in the (η, f')-plane. We observe a strong decrease in the amplitude of the characteristic oscillation of f' around -1 as λ approaches 2.

In order to catch hold of the bifurcation process described above we have tried to find an approximation to $f' + 1$ in regions where $f' \approx -1$ by putting

$$f(\eta) = -\eta + \epsilon(\eta) \tag{5}$$

where ϵ is supposed to be small and η has been taken such that again $f(0) = f''(0) = 0$,

Table 1. The period and $f'(0)$ for the periodic solutions P-i when $\lambda \downarrow i$; $i = 1, 2, 3$

P-1			P-2			P-3		
λ	$f'(0)$	period	λ	$f'(0)$	period	λ	$f'(0)$	period
2	0.916911	7.03	3	0.994099	12.20	4	0.999556	16.17
1.64	0.917696	8.30	2.64	0.994401	13.93	3.64	0.999594	18.03
1.32	0.925404	10.62	2.32	0.995436	16.82	3.32	0.999691	20.93
1.16	0.939754	13.60	2.16	0.996713	20.01	3.16	0.999789	23.83
1.08	0.956901	17.75	2.08	0.997836	23.60	3.08	0.999866	26.79
1.04	0.972467	23.77	2.04	0.998665	27.71	3.04	0.999918	29.89
1.02	0.983916	32.56	2.02	0.999215	32.52	3.02	0.999951	33.23
1.01	0.991167	45.25	2.01	0.999556	38.20	3.01	0.999972	36.90
(1	1	∞)	(2	1	∞)	(3	1	∞)

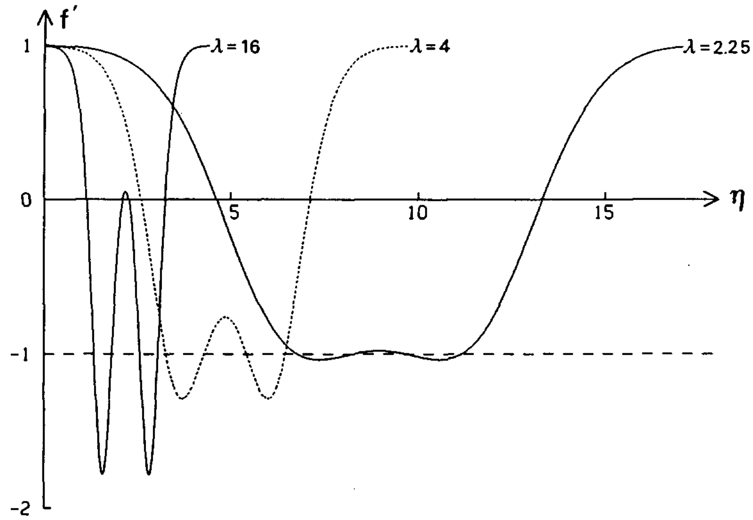

Figure 5. The periodic solution P-2 for $\lambda = 16$, 4 and 2.25.

but now with $f'(0) \approx -1$. After substituting (5) into (1) and linearization with respect to ϵ we get

$$\epsilon''' - \eta\epsilon'' + 2\lambda\epsilon' = 0. \tag{6}$$

Taking $\epsilon' = y(\eta)\exp(\frac{1}{4}\eta^2)$ yields

$$y'' - \left(\tfrac{1}{4}\eta^2 - 2\lambda - \tfrac{1}{2}\right)y = 0. \tag{7}$$

Since ϵ' is even, y is even too, and the solution of (7) is given by

$$y = C\exp\left(-\tfrac{1}{4}\eta^2\right)M\left(-\lambda, \tfrac{1}{2}, \tfrac{1}{2}\eta^2\right),$$

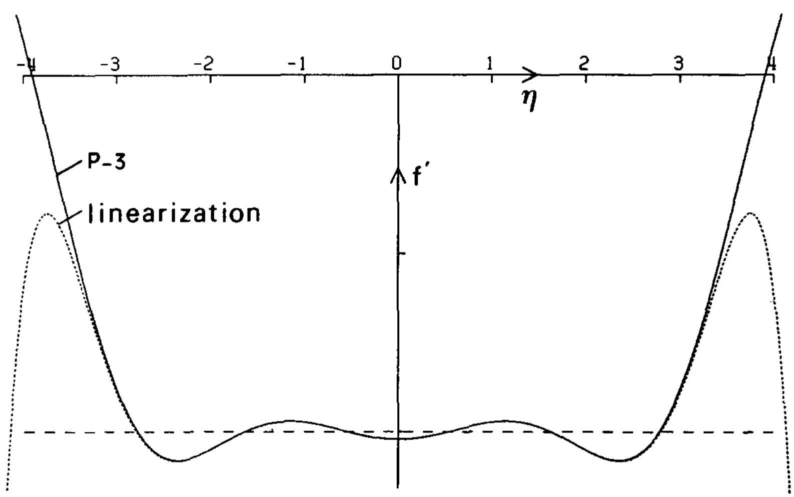

Figure 6. Comparison between P-3 and the linearization.

see Abramowitz and Stegun [10]. Thus, the approximation of $f' + 1$ $(=\epsilon')$ we are looking for is given by

$$f' + 1 \approx CM\left(-\lambda, \tfrac{1}{2}, \tfrac{1}{2}\eta^2\right) \tag{8}$$

where we take the constant C such that both sides are equal for $\eta = 0$. For $\lambda = 4$ and the periodic solution P-3 we see in Fig. 6 how well both sides of (8) match. Moreover, the confluent hypergeometric function M in (8) has the property that it has exactly $2n + 2$ zeroes (Erdélyi [11]), with n the positive integer satisfying $n < \lambda \leqslant n + 1$. Thus every time λ passes a positive integer i from below, the number of zeros of M increases by two. This is in perfect agreement with our conjecture that at the same moment a new periodic solution P-i is formed, following one more oscillation of M.

4. A classification of solutions and further results

When we consider Fig. 2 again, it seems quite natural to denote the various solutions by the type of the successive periodic solutions followed approximately. In this way the first four Falkner-Skan solutions (a–d) are denoted by F, F-1, F-1-1 and F-1-1-1, respectively, and the solutions (e–h) by F-2, F-2-2, F-1-2 and F-3, respectively. Especially the appearance of F-1-2 raises the following question: can any combination of available periodic types be made for a given value of λ? In answering this question we first tried to calculate some other combinations. However, it will be clear that for the calculation of combinations of the P-i the integration has to be performed sometimes over an extremely large interval, especially for the combinations with relatively large values of i. Therefore a simple shooting method becomes impracticable.

This difficulty can be met by using a multiple shooting method, where the integration interval is subdivided into a number of subintervals. Let us describe this method very briefly for our problem (1) + (2) and the simple case of two subintervals $[0, \eta_1]$ and $[\eta_1, \eta_2]$, with η_2 large enough to take $f'(\eta_2) = 1$. With starting values $f^{(i)}(\eta_1) = x_i$, $i = 0, 1, 2$, and $f''(0) = x_3$ we can integrate (1) over both subintervals separately. The condition for continuity in η_1 and the remaining boundary condition in η_2 lead to four nonlinear equations in the x_i. These equations are solved by a Newton method, where the Jacobian matrix can be formed by integrating related initial-value problems. For further details we refer to Cebeci and Keller [12].

It is well known that in general the convergence of Newton's method is quadratic, assuming that the starting values are chosen sufficiently close to the exact solution. In general the need for accurate starting values is a serious drawback, but here we can exploit the knowledge about the periodic solutions and the solutions of type F-1, F-2, . . . , already found by a simple shooting method. The starting values can be obtained easily from the "prescribed" periodic solutions. Thereafter, a sufficiently large "tail" is attached where $f' = 1^-$ and f'' is slightly positive. With this strategy the multiple shooting method turned out to work quite satisfactorily and each case showed perfect quadratic convergence. For $\lambda = 4$ some of the results are given in Table 2.

Some care has to be taken in choosing the length of the subintervals in the multiple shooting process. If these intervals are taken too large, the profits of the multiple shooting algorithm are not fully exploited; the integration process will still be the bottleneck. If the subintervals are taken too small, the dimension of the problem becomes large. This

Table 2. Various Falkner-Skan solutions for $\lambda = 4$.

Type	$f''(0)$	Type	$f''(0)$
F	2.347285955749827690	F-1-1	2.231902851594374559
F-3-1	2.346603756525838409	F-1-1-3	2.231902847881036270
F-3-1-1	2.346603756525835595	F-1-1-2-1	2.231902804102634407
F-3-2	2.346603756524728496	F-1-1-2-2	2.231902804102235083
F-3-2-1	2.346603756524728495	F-1-1-2	2.231902804102202213
F-3	2.346603756524637163	F-1-1-1-1	2.231902229118067373
F-2-1	2.338567516379669706	F-1-1-1-2	2.231902227772409051
F-2-1-3	2.338567516378565330	F-1-1-1	2.231902227661542085
F-2-1-2	2.338567516365545167	F-1-2	2.231655937870309033
F-2-1-1	2.338567516194108350	F-1-2-2-1	2.231655937856229689
F-2-2	2.338567443148880241	F-1-2-2	2.231655937856229560
F-2-2-3	2.338567443148879915	F-1-2-1-1	2.231655937685618276
F-2-2-2	2.338567443148876068	F-1-2-1-2	2.231655937685217852
F-2-2-1	2.338567443148825372	F-1-2-1	2.231655937685184862
F-2-3	2.338567437592141701	F-1-3	2.231637185921316231
F-2-3-2	2.338567437592141701	F-1-3-2	2.231637185921316018
F-2-3-1	2.338567437592141700	F-1-3-1	2.231637185921313430
F-2	2.338567437120918718	F-1	2.231635595381601265

hampers the convergence of the Newton process. For $\lambda = 4$ an interval length of 1.5–2 is satisfactory.

If there are solutions F-1-2, F-1-2-1-2, F-1-2-1-2-1-2, etc., one expects that there will also be a periodic solution which approximately follows one period of P-1 and subsequently one period of P-2. Such a periodic solution has indeed been found and will be denoted by P-1-2. For the calculation we used again the multiple shooting method, slightly modified because of the conditions for periodicity. Further results even suggest a similar classification for the periodic solutions of (1) as has been introduced for the Falkner-Skan solutions. Figure 7 shows as an example the periodic solution P-1-2-3 in the (f, f')-plane.

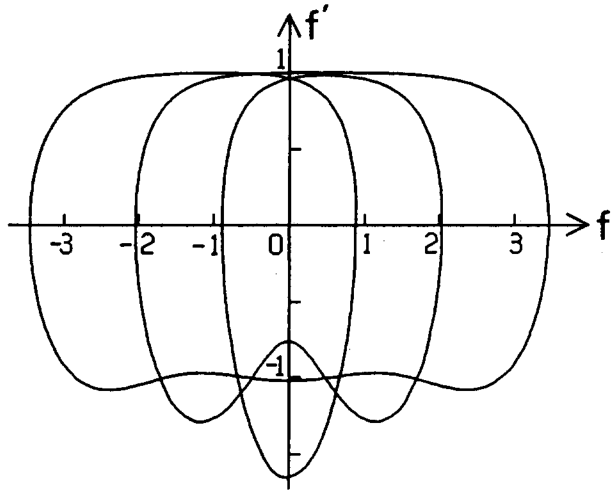

Figure 7. The periodic solution P-1-2-3 for $\lambda = 4$.

We remark that, for example, P-1-2 and P-3 cannot be distinguished by the number of (relative) minima of f' in a period. An essential difference, however, is that P-1-2 exists for $\lambda > 2$, whereas P-3 exists for $\lambda > 3$.

5. Solutions for various λ

The solutions given in Table 2 have been ordered with respect to $f''(0)$. For F this value is greater than for any other solution of (1) + (2); a proof is given in the appendix. If we denote the value $f''(0)$ of the solution F-i by τ(F-i), the following orderings can be inferred:

$$\tau(\text{F-1}) < \tau(\text{F-2}) < \tau(\text{F-3}) < \ldots < \tau(\text{F}), \tag{9}$$

$$\tau(\text{F-1-1}) > \tau(\text{F-1-2}) > \tau(\text{F-1-3}) > \ldots > \tau(\text{F-1}), \tag{10}$$

$$\tau(\text{F-1-1-1}) < \tau(\text{F-1-1-2}) < \tau(\text{F-1-1-3}) < \ldots < \tau(\text{F-1-1}). \tag{11}$$

The general pattern appears to be that we may replace F in (9) with an F followed by an even number of integers, e.g. replacement of F with F-1-1 leads to (11). If we replace F in (9) with an F followed by an odd number of integers, the inequality signs must be reversed; (10) is an example.

The results of Table 2 have been extended to other values of λ by combining the multiple shooting algorithm with a continuation method. Figure 8 gives a schematic sketch of the bifurcation pattern in the (λ, $f''(0)$)-plane. The classical monotonic Falkner-Skan solution gives the main branch, indicated by F. For each integer value of λ a bifurcation takes place, in which a bundle of additional solution branches separates from the main branch. In Fig. 8 we have indicated the limiting branches for each bundle; the distance between the branches is highly exaggerated. At $\lambda = 1$ the bundle corresponding with the Craven and Peletier solutions emanates. The limiting branches are F-1 and F-1-1. At $\lambda = 2$ another bundle separates from the main branch. Since now two periodic solutions exist, the structure inside this bundle is more complicated. It consists of solutions F-2, F-2-i_1, F-2-i_1-i_2, . . . , with the $i_j \in \{1, 2\}$. Limiting branches are F-2 and F-2-1.

At larger integer values of λ a similar bifurcation process occurs along the main branch. Its complexity increases with the number of available periodic solutions. The bundle starting at $\lambda = k$ is bounded by the solutions F-k and F-k-1, and consists of solutions F-k, F-k-i_1, F-k-i_1-i_2, . . . , with the $i_j \in \{1, 2, \ldots, k\}$.

Also inside each bundle bifurcation occurs when λ passes an integer value. This bifurcation proceeds in the same way as the bifurcation of the main branch, although the ordering of the branches with respect to $f''(0)$ can be reversed. Starting for $\lambda = 1$ on F and following the branches, the general pattern seems to be that a solution branch F-i_1-. . .-i_n bifurcates at its right-hand side when n is even and at its left-hand side when n is odd. For instance, in the Craven and Peletier bundle at $\lambda = 2$, the solution branch F-1 gives birth to a bundle bounded by F-1-2 and F-1-2-1, lying above the original branch. Similarly, from the branch F-1-1, a bundle bounded by F-1-1-2 and F-1-1-2-1 springs off, lying below F-1-1. In this way the whole bifurcation process in each bundle takes place between its limiting branches.

The bundles with solution branches can be followed for increasing λ. For moderate values of λ they lie close to the main branch, but eventually they bend away, pass a

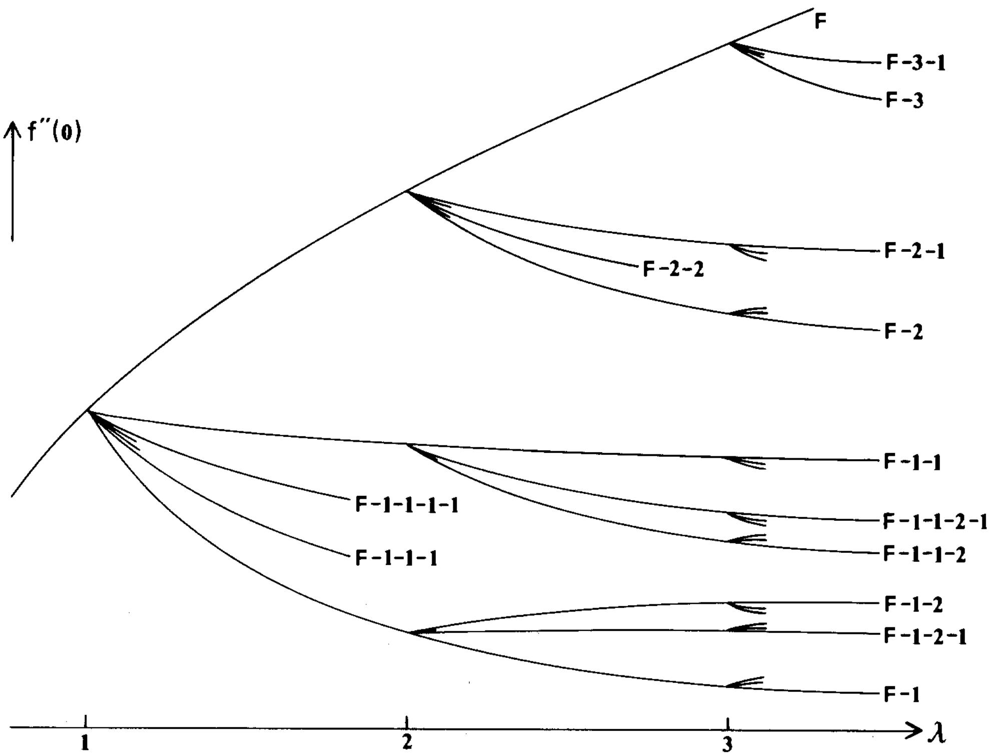

Figure 8. Schematic sketch of the bifurcation pattern.

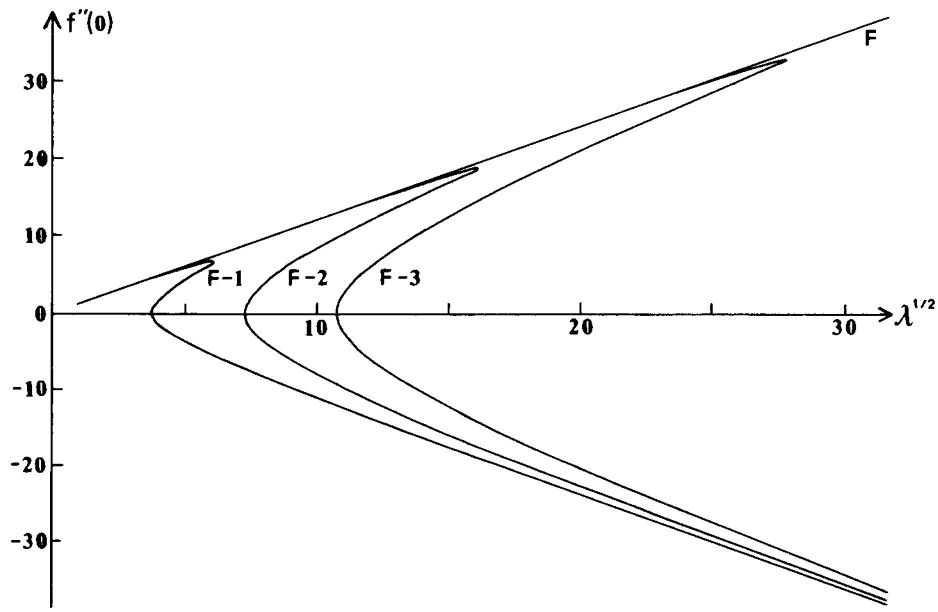

Figure 9. Location of solutions in the $(\lambda^{1/2}, f''(0))$-plane.

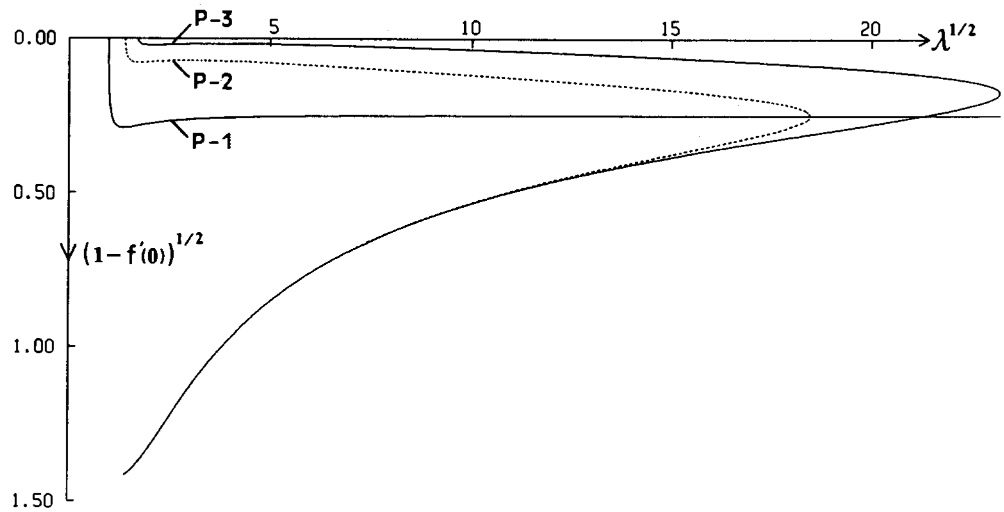

Figure 10. Continuation of the value $f'(0)$ for P-1, P-2 and P-3.

turning point, and proceed in the direction of decreasing λ. At the intersection of these bundles with the λ-axis another turning point is located. Hereafter the bundle runs again in positive λ-direction. Figure 9 indicates the location of the bundles starting at $\lambda = 1$, 2 and 3 respectively, by plotting one of their limiting branches, namely F-1, F-2 and F-3. The bundles are extremely narrow as can be inferred from Table 2.

We also followed the periodic solutions P-1, P-2 and P-3 for increasing λ, starting from 1, 2 and 3, respectively. As in Table 1, we characterized these periodic solutions by the largest value of f' in the two points where $f = f'' = 0$, thus starting with $f' = 1^-$. The results – after a suitable stretching – are shown in Fig. 10. The curve for P-1 tends to a horizontal asymptote. The curve for P-2 has a turning point for $\lambda \approx 340$, where P-2 coincides with two periods of P-1. The start of the deformation of P-2 can be followed in

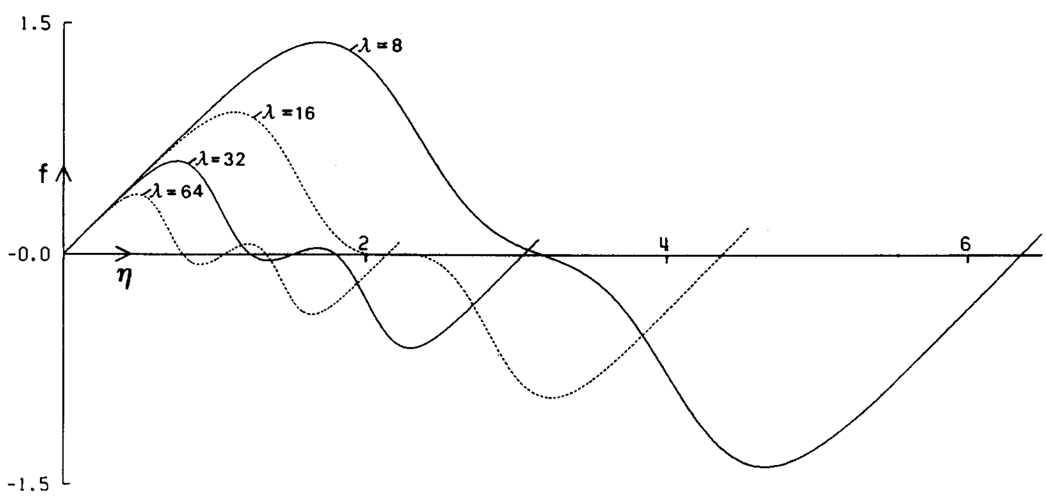

Figure 11. Deformation of P-2 for increasing λ.

Fig. 11, where P-2 is given for $\lambda = 8$, 16, 32 and 64. The lower half of the curve for P-2 corresponds with the value of f' in the other point where $f = f'' = 0$. The curve for P-3 turns back for $\lambda \approx 538$. Thereafter it intersects the curve for P-1 and at this point P-3 coincides with three periods of P-1. The curve has been followed for decreasing λ until $\lambda = 2^+$, where the corresponding periodic solution is of the type P-1-2. At the end of Section 4 we already noticed the close relationship between P-3 and P-1-2.

By the time this investigation was completed, a paper by Hastings and Troy [13] appeared on the same subject. They found the same types of periodic solutions; however, their conjecture that these solutions exist for all (sufficiently large) λ is believed to be incorrect (see Fig. 10). This may be the reason why Hastings and Troy were not able to prove their conjecture for large λ. Further, they conjectured the existence of solutions of (1) + (2) composed of arbitrarily ordered periodic solutions. The computations described in the present paper show that this is indeed very likely.

6. Conclusions

The Falkner-Skan equation for $\lambda > 1$ gives rise to a multitude of solutions. By numerical means, a complicated branching structure has been unraveled, with bifurcation occurring at every positive integer value of λ.

An important role is played by the antisymmetric periodic solutions P-i, which are characterized by confluent zeros of f and f''. They are distinguished by i, the number of relative minima of f' in a period. Such a periodic solution exists from $\lambda = i$ up to, presumably, a finite value of λ.

For $\lambda > 2$ more general periodic solutions have been found, which can be considered to be composed of arbitrary sequences of the periodic solutions P-i.

In an analogous fashion the branches of the Falkner-Skan solutions can be classified. It is conjectured that these branches exist up to $\lambda = \infty$. Their asymptotic behaviour is currently under investigation.

The Falkner-Skan equation yields an exact solution of the equations describing boundary-layer flow. There are more situations in fluid dynamics where chaotic solutions are known to exist [14]. Further study of the relatively simple Falkner-Skan equation may help to bring order into this chaos.

Appendix

LEMMA: *Let $f(\eta)$ denote the Falkner-Skan solution F of* (1) + (2) *for* $\lambda > 0$, *then all other solutions $g(\eta)$ of* (1) + (2) *must satisfy* $g''(0) < f''(0)$.

PROOF: From [4] and [5] we already know that the problem (1) + (2) has a unique solution for $0 \leqslant \lambda \leqslant 1$, thus we only have to consider $\lambda > 1$. As f represents the unique monotonic solution, c.f. Hartman [6], we have for $\eta > 0$: $0 < f' < 1$, $f'' > 0$, $f''' < 0$. Let $g(\eta)$ be a solution of (1) + (2) with $f''(0) < g''(0)$ and take $v(\eta) = g(\eta) - f(\eta)$, then

$$v(0) = v'(0) = 0, \quad v''(0) > 0.$$

Since $v'(\infty) = 0$, there must be a point p where v' has its first (relative) maximum, hence

$$v'(p) > 0, \quad v''(p) = 0, \quad v'''(p) \leqslant 0. \tag{A1}$$

Substitution of $g=f+v$ in (1) yields for $\eta=p$:

$$v'''(p)=2\lambda f'(p)v'(p)-f''(p)v(p)+\lambda(v'(p))^2. \tag{A2}$$

From $f''' < 0$ it readily follows that

$$f'(p)>pf''(p), \tag{A3}$$

and, since $v''>0$ for $0 \leqslant \eta < p$, we can derive similarly that

$$v(p)<pv'(p). \tag{A4}$$

Using (A3) and (A4) in (A2) yields

$$v'''(p)>(2\lambda-1)pf''(p)v'(p)+\lambda(v'(p))^2>0,$$

which violates (A1).

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